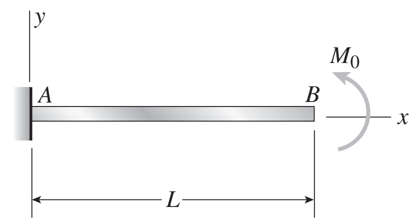
Deflections by Integration of the Shear Force and Load Equations

The beams described in the problems for Section 9.4 have constant flexural rigidity EI. Also, the origin of coordinates is at the left-hand end of each beam.

Problem 9.4-1 Derive the equation of the deflection curve for a cantilever beam AB when a couple M_0 acts counterclockwise at the free end (see figure). Also, determine the deflection δ_B and slope θ_B at the free end. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-1 Cantilever beam (couple M_0)

SHEAR-FORCE EQUATION (EQ. 9-12 b).

$$EIv''' = V = 0$$

$$EIv'' = C_1$$

B.C. 1 $M = M_0 \quad EIv'' = M = M_0 = C_1$

$$EIv' = C_1 x + C_2 = M_0 x + C_2$$

B.C. 2 $v'(0) = 0 \quad \therefore C_2 = 0$

$$EIv = \frac{M_0 x^2}{2} + C_3$$

B.C. 3 $v(0) = 0 \quad \therefore C_3 = 0$

$$v = \frac{M_0 x^2}{2EI} \leftarrow$$

$$v' = \frac{M_0 x}{EI}$$

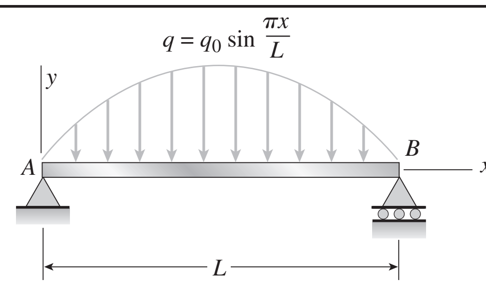
$$\delta_B = v(L) = \frac{M_0 L^2}{2EI} \text{ (upward)} \leftarrow$$

$$\theta_B = v'(L) = \frac{M_0 L}{EI} \text{ (counterclockwise)} \leftarrow$$

(These results agree with Case 6, Table G-1.)

Problem 9.4-2 A simple beam AB is subjected to a distributed load of intensity $q = q_0 \sin \pi x/L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_{max} at the midpoint of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-2 Simple beam (sine load)

LOAD EQUATION (EQ. 9-12 c).

$$EIv'''' = -q = -q_0 \sin \frac{\pi x}{L}$$

$$EIv''' = q_0 \left(\frac{L}{\pi}\right) \cos \frac{\pi x}{L} + C_1$$

$$EIv'' = q_0 \left(\frac{L}{\pi}\right)^2 \sin \frac{\pi x}{L} + C_1 x + C_2$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = 0$

$$EIv' = -q_0 \left(\frac{L}{\pi}\right)^3 \cos \frac{\pi x}{L} + C_3$$

$$EIv = -q_0 \left(\frac{L}{\pi}\right)^4 \sin \frac{\pi x}{L} + C_3 x + C_4$$

B.C. 3 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_3 = 0$

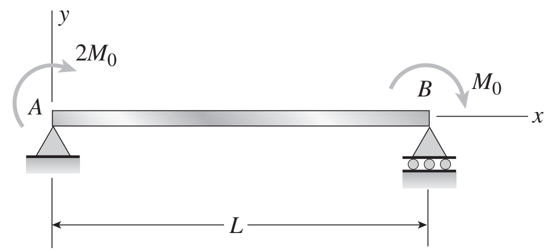
$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L} \leftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{q_0 L^4}{\pi^4 EI} \leftarrow$$

(These results agree with Case 13, Table G-2.)

Problem 9.4-3 The simple beam AB shown in the figure has moments $2M_0$ and M_0 acting at the ends.

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{max}. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-3 Simple beam with two couples

Reaction at support A: $R_A = \dfrac{3M_0}{L}$ (downward)

Shear force in beam: $V = -R_A = -\dfrac{3M_0}{L}$

SHEAR-FORCE EQUATION (EQ. 9-12 b)

$$EIv''' = V = -\frac{3M_0}{L}$$

$$EIv'' = -\frac{3M_0 x}{L} + C_1$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 2M_0 \quad \therefore C_1 = 2M_0$

$$EIv' = -\frac{3M_0 x^2}{2L} + 2M_0 x + C_2$$

$$EIv = -\frac{M_0 x^3}{2L} + M_0 x^2 + C_2 x + C_3$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

B.C. 3 $v(L) = 0 \quad \therefore C_2 = -\dfrac{M_0 L}{2}$

$$v = -\frac{M_0 x}{2LEI}(L^2 - 2Lx + x^2) = -\frac{M_0 x}{2LEI}(L - x)^2 \longleftarrow$$

$$v' = -\frac{M_0}{2LEI}(L - x)(L - 3x)$$

MAXIMUM DEFLECTION

Set $v' = 0$ and solve for x:

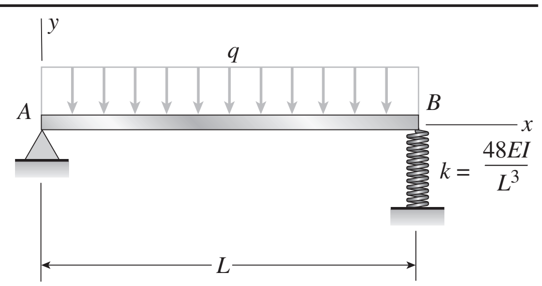
$x_1 = L$ and $x_2 = \dfrac{L}{3}$

Maximum deflection occurs at $x_2 = \dfrac{L}{3}$.

$$\delta_{max} = -v\left(\frac{L}{3}\right) = \frac{2M_0 L^2}{27EI} \text{ (downward)} \longleftarrow$$

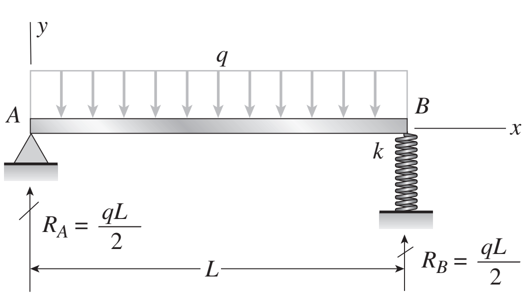
Problem 9.4-4 A simple beam with a uniform load is pin supported at one end and spring supported at the other. The spring has stiffness $k = 48EI/L^3$.

Derive the equation of the deflection curve by starting with the third-order differential equation (the shear-force equation). Also, determine the angle of rotation θ_A at support A.



Solution 9.4-4 Beam with a spring support

REACTIONS



DEFLECTION AT END B

$$k = \frac{48EI}{L^3} \qquad \delta_B = \frac{R_B}{k} = \frac{qL}{2k} = \frac{qL^4}{96EI}$$

SHEAR-FORCE EQUATION (EQ. 9-12 b)

$$V = R_A - qx = \frac{q}{2}(L - 2x)$$

$$EIv''' = V = \frac{q}{2}(L - 2x)$$

$$EIv'' = \frac{q}{2}(Lx - x^2) + C_1$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_1 = 0$

$$EIv' = \frac{q}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_2$$

$$EIv = \frac{q}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_2 x + C_3$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

B.C. 3 $v(L) = -\delta_B = -\dfrac{qL^4}{96EI}$

$$\therefore C_2 = -\frac{5qL^3}{96}$$

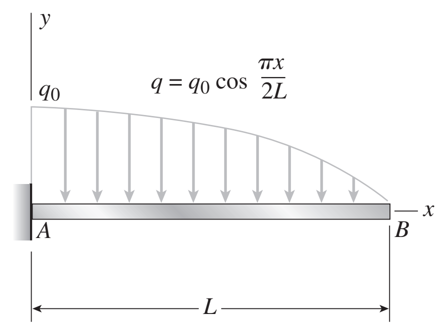
$$v = -\frac{qx}{96EI}(5L^3 - 8Lx^2 + 4x^3) \longleftarrow$$

$$v' = -\frac{q}{96EI}(5L^3 - 24Lx^2 + 16x^3)$$

$$\theta_A = -v'(0) = \frac{5qL^3}{96EI} \quad \text{(clockwise)} \longleftarrow$$

Problem 9.4-5 The distributed load acting on a cantilever beam AB has an intensity q given by the expression $q_0 \cos \pi x/2L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-5 Cantilever beam (cosine load)

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -q_0 \cos\frac{\pi x}{2L}$$

$$EIv''' = -q_0\left(\frac{2L}{\pi}\right)\sin\frac{\pi x}{2L} + C_1$$

B.C. 1 $EIv''' = V \quad EIv'''(L) = 0 \quad \therefore C_1 = \dfrac{2q_0 L}{\pi}$

$$EIv'' = q_0\left(\frac{2L}{\pi}\right)^2\cos\frac{\pi x}{2L} + \frac{2q_0 Lx}{\pi} + C_2$$

B.C. 2 $EIv'' = M \quad EIv''(L) = 0 \quad \therefore C_2 = -\dfrac{2q_0 L^2}{\pi}$

$$EIv' = q_0\left(\frac{2L}{\pi}\right)^3\sin\frac{\pi x}{2L} + \frac{q_0 Lx^2}{\pi} - \frac{2q_0 L^2 x}{\pi} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -q_0\left(\frac{2L}{\pi}\right)^4\cos\frac{\pi x}{2L} + \frac{q_0 Lx^3}{3\pi} - \frac{q_0 L^2 x^2}{\pi} + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = \dfrac{16q_0 L^4}{\pi^4}$

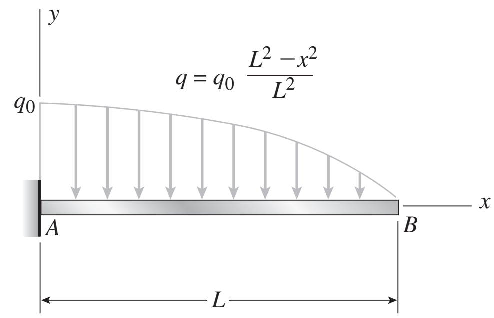
$$v = -\frac{q_0 L}{3\pi^4 EI}\left(48L^3\cos\frac{\pi x}{2L} - 48L^3 + 3\pi^3 Lx^2 - \pi^3 x^3\right) \longleftarrow$$

$$\delta_B = -v(L) = \frac{2q_0 L^4}{3\pi^4 EI}(\pi^3 - 24) \longleftarrow$$

(These results agree with Case 10, Table G-1.)

Problem 9.4-6 A cantilever beam AB is subjected to a parabolically varying load of intensity $q = q_0(L^2 - x^2)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B and angle of rotation θ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-6 **Cantilever beam (parabolic load)**

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{q_0}{L^2}(L^2 - x^2)$$

$$EIv''' = -\frac{q_0}{L^2}\left(L^2x - \frac{x^3}{3}\right) + C_1$$

B.C. 1 $EIv''' = V \quad EIv'''(L) = 0 \quad \therefore C_1 = \dfrac{2q_0L}{3}$

$$EIv'' = -\frac{q_0}{L^2}\left(\frac{L^2x^2}{2} - \frac{x^4}{12}\right) + \frac{2q_0L}{3}x + C_2$$

B.C. 2 $EIv'' = M \quad EIv''(L) = 0 \quad \therefore C_2 = -\dfrac{q_0L^2}{4}$

$$EIv' = -\frac{q_0}{L^2}\left(\frac{L^2x^3}{6} - \frac{x^5}{60}\right) + \frac{q_0Lx^2}{3} - \frac{q_0L^2x}{4} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{q_0}{L^2}\left(\frac{L^2x^4}{24} - \frac{x^6}{360}\right) + \frac{q_0Lx^3}{9} - \frac{q_0L^2x^2}{8} + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = 0$

$$v = -\frac{q_0x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4) \longleftarrow$$

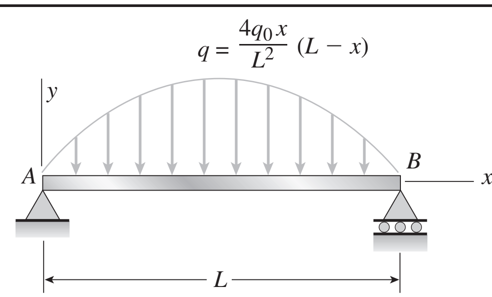
$$\delta_B = -v(L) = \frac{19q_0L^4}{360EI} \longleftarrow$$

$$v' = -\frac{q_0x}{60L^2EI}(15L^4 - 20L^3x + 10L^2x^2 - x^4)$$

$$\theta_B = -v'(L) = \frac{q_0L^3}{15EI} \longleftarrow$$

Problem 9.4-7 A beam on simple supports is subjected to a parabolically distributed load of intensity $q = 4q_0x(L - x)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{max}. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-7 **Simple beam (parabolic load)**

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{4q_0x}{L^2}(L - x) = -\frac{4q_0}{L^2}(Lx - x^2)$$

$$EIv''' = -\frac{2q_0}{3L^2}(3Lx^2 - 2x^3) + C_1$$

$$EIv'' = -\frac{q_0}{3L^2}(2Lx^3 - x^4) + C_1x + C_2$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = \dfrac{q_0L}{3}$

$$EIv' = -\frac{q_0}{30L^2}(-5L^3x^2 + 5Lx^4 - 2x^5) + C_3$$

B.C. 3 (Symmetry) $v'\left(\dfrac{L}{2}\right) = 0 \quad \therefore C_3 = -\dfrac{q_0L^3}{30}$

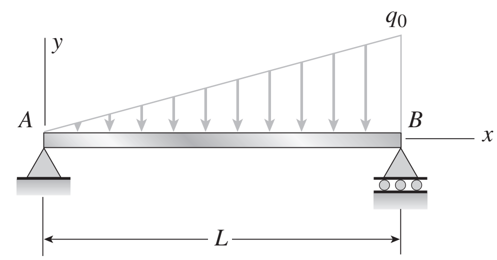
$$EIv = -\frac{q_0}{30L^2}\left(L^5x - \frac{5L^3x^3}{3} + Lx^5 - \frac{x^6}{3}\right) + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = 0$

$$v = -\frac{q_0x}{90L^2EI}(3L^5 - 5L^3x^2 + 3Lx^4 - x^5) \longleftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{61q_0L^4}{5760EI} \longleftarrow$$

Problem 9.4-8 Derive the equation of the deflection curve for a simple beam *AB* carrying a triangularly distributed load of maximum intensity q_0 (see figure). Also, determine the maximum deflection δ_{max} of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-8 Simple beam (triangular load)

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{q_0 x}{L} \qquad EIv''' = -\frac{q_0 x^2}{2L} + C_1$$

$$EIv'' = -\frac{q_0 x^3}{6L} + C_1 x + C_2$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = \frac{q_0 L}{6}$

$$EIv' = -\frac{q_0 x^4}{24L} + \frac{q_0 L x^2}{12} + C_3$$

$$EIv = -\frac{q_0 x^5}{120L} + \frac{q_0 L x^3}{36} + C_3 x + C_4$$

B.C. 3 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_3 = -\frac{7q_0 L^3}{360}$

$$v = -\frac{q_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4) \longleftarrow$$

$$v' = -\frac{q_0}{360LEI}(7L^4 - 30L^2x^2 + 15x^4)$$

MAXIMUM DEFLECTION

Set $v' = 0$ and solve for x:

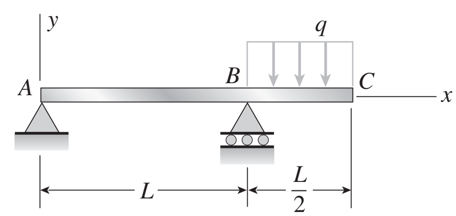
$$x_1^2 = L^2\left(1 - \sqrt{\frac{8}{15}}\right) \qquad x_1 = 0.51933L$$

$$\delta_{max} = -v(x_1) = \frac{q_0 L^4}{225EI}\left(\frac{5}{3} + \frac{2}{3}\sqrt{\frac{8}{15}}\right)^{1/2} \longleftarrow$$

$$= 0.006522\frac{q_0 L^4}{EI} \longleftarrow$$

(These results agree with Case 11, Table G-2.)

Problem 9.4-9 Derive the equations of the deflection curve for an overhanging beam *ABC* subjected to a uniform load of intensity q acting on the overhang (see figure). Also, obtain formulas for the deflection δ_C and angle of rotation θ_C at the end of the overhang. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-9 Beam with an overhang

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = 0 \qquad (0 \le x \le L)$$
$$EIv''' = C_1 \qquad (0 \le x \le L)$$
$$EIv'' = C_1 x + C_2 \qquad (0 \le x \le L)$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

$$EIv'''' = -q \qquad \left(L \le x \le \frac{3L}{2}\right)$$

$$EIv''' = -qx + C_3 \qquad \left(L \le x \le \frac{3L}{2}\right)$$

B.C. 2 $EIv''' = V \quad EIv'''\left(\frac{3L}{2}\right) = 0 \quad \therefore C_3 = \frac{3qL}{2}$

$$EIv'' = -\frac{qx^2}{2} + \frac{3qLx}{2} + C_4 \quad \left(L \le x \le \frac{3L}{2}\right)$$

B.C. 3 $EIv'' = M \quad EIv''\left(\frac{3L}{2}\right) = 0 \quad \therefore C_4 = -\frac{9qL^2}{8}$

B.C. 4 $EI(v'')_{Left} = EI(v'')_{Right}$ at $x = L$

$$C_1 L = -\frac{qL^2}{2} + \frac{3qL^2}{2} - \frac{9qL^2}{8} \qquad \therefore C_1 = -\frac{qL}{8}$$

$$EIv' = -\frac{qLx^2}{16} + C_5 \quad (0 \le x \le L)$$

$$EIv' = -\frac{qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8} + C_6$$

$$\left(L \le x \le \frac{3L}{2}\right)$$

(Continued)

B.C. 5 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = L$

$$\therefore C_6 = C_5 + \frac{23qL^3}{48} \qquad \text{(a)}$$

$$EIv = -\frac{qLx^3}{48} + C_5 x + C_7 \quad (0 \le x \le L)$$

B.C. 6 $v(0) = 0 \quad \therefore C_7 = 0$

B.C. 7 $v(L) = 0$ for $0 \le x \le L \quad \therefore C_5 = \frac{qL^3}{48}$

From Eq.(a): $C_6 = \frac{qL^3}{2}$

$$EIv = -\frac{qx^4}{24} + \frac{3qLx^3}{12} - \frac{9qL^2x^2}{16} + \frac{qL^3x}{2} + C_8 \quad \left(L \le x \le \frac{3L}{2}\right)$$

B.C. 8 $v(L) = 0$ for $L \le x \le \frac{3L}{2} \quad \therefore C_8 = -\frac{7qL^4}{48}$

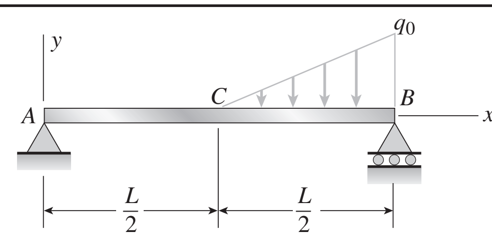
$$v = \frac{qLx}{48\,EI}(L^2 - x^2) \quad (0 \le x \le L) \quad \longleftarrow$$

$$v = -\frac{q(L-x)}{48\,EI}(7L^3 - 17L^2x + 10Lx^2 - 2x^3) \quad \longleftarrow \quad \left(L \le x \le \frac{3L}{2}\right)$$

$$\delta_C = -v\left(\frac{3L}{2}\right) = \frac{11qL^4}{384\,EI} \quad \longleftarrow$$

$$\theta_C = -v'\left(\frac{3L}{2}\right) = \frac{qL^3}{16\,EI} \quad \longleftarrow$$

Problem 9.4-10 Derive the equations of the deflection curve for a simple beam AB supporting a triangularly distributed load of maximum intensity q_0 acting on the right-hand half of the beam (see figure). Also, determine the angles of rotation θ_A and θ_B at the ends and the deflection δ_C at the midpoint. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-10 **Simple beam (triangular load)**

LOAD EQUATION (EQ. 9-12 c)

Left-hand half (part AC): $0 \le x \le \frac{L}{2}$

Right-hand half (part CB): $\frac{L}{2} \le x \le L$

PART AC $\quad q = 0$

$$EIv'''' = -q = 0 \qquad EIv''' = C_1$$

$$EIv'' = C_1 x + C_2 \quad EIv' = C_1\left(\frac{x^2}{2}\right) + C_2 x + C_3$$

$$EIv = C_1\left(\frac{x^3}{6}\right) + C_2\left(\frac{x^2}{2}\right) + C_3 x + C_4$$

PART CB $\quad q = \frac{q_0}{L}(2x - L)$

$$EIv'''' = -q = \frac{q_0}{L}(L - 2x)$$

$$EIv''' = \frac{q_0}{L}(Lx - x^2) + C_5$$

$$EIv'' = \frac{q_0}{L}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_5 x + C_6$$

$$EIv' = \frac{q_0}{L}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_5\left(\frac{x^2}{2}\right) + C_6 x + C_7$$

$$EIv = \frac{q_0}{L}\left(\frac{Lx^4}{24} - \frac{x^5}{60}\right) + C_5\left(\frac{x^3}{6}\right) + C_6\left(\frac{x^2}{2}\right) + C_7 x + C_8$$

BOUNDARY CONDITIONS

B.C. 1 $EIv''' = V \quad EI(v''')_{AC} = EI(v''')_{BC}$ at $x = \frac{L}{2}$

$$C_1 - C_5 = \frac{q_0 L}{4} \qquad (1)$$

B.C. 2 $EIv'' = M \quad EIv''(0) = 0$

$$C_2 = 0 \qquad (2)$$

B.C. 3 $EIv''(L) = 0 \quad C_5 L + C_6 = -\frac{q_0 L^2}{6} \qquad (3)$

B.C. 4 $(EIv'')_{AC} = (EIv'')_{CB}$ for $x = \frac{L}{2}$

$$C_1 L - C_5 L - 2C_6 = \frac{q_0 L^2}{6} \qquad (4)$$

B.C. 5 $(v')_{AC} = (v')_{CB}$ for $x = \frac{L}{2}$

$$C_1L^2 + 8C_3 - C_5L^2 - 4C_6L - 8C_7 = \frac{q_0L^3}{8} \quad (5)$$

B.C. 6 $v(0) = 0 \quad C_4 = 0 \quad (6)$

B.C. 7 $v(L) = 0$

$$C_5L^3 + 3C_6L^2 + 6C_7L + 6C_8 = -\frac{3q_0L^4}{20} \quad (7)$$

B.C. 8 $(v)_{AC} = (v)_{CB}$ for $x = \frac{L}{2}$

$$C_1L^3 + 24C_3L - C_5L^3 - 6C_6L^2 - 24C_7L - 48C_8 = \frac{q_0L^4}{10} \quad (8)$$

SOLVE EQS. (1) THROUGH (8):

$$C_1 = \frac{q_0L}{24} \quad C_2 = 0 \quad C_3 = -\frac{37q_0L^3}{5760}$$

$$C_4 = 0 \quad C_5 = -\frac{5q_0L}{24} \quad C_6 = \frac{q_0L^2}{24}$$

$$C_7 = -\frac{67q_0L^3}{5760} \quad C_8 = \frac{q_0L^4}{1920}$$

Substitute constants into equations for v and v'.

DEFLECTION CURVE FOR PART $AC \left(0 \le x \le \frac{L}{2}\right)$

$$v = -\frac{q_0Lx}{5760EI}(37L^2 - 40x^2) \quad \leftarrow$$

$$v' = -\frac{q_0L}{5760EI}(37L^2 - 120x^2)$$

$$\theta_A = -v'(0) = \frac{37q_0L^3}{5760EI} \quad \leftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{3q_0L^4}{1280EI} \quad \leftarrow$$

DEFLECTION CURVE FOR PART $CB \left(\frac{L}{2} \le x \le L\right)$

$$v = -\frac{q_0}{5760LEI}[L^2x(37L^2 - 40x^2) + 3(2x - L)^5] \quad \leftarrow$$

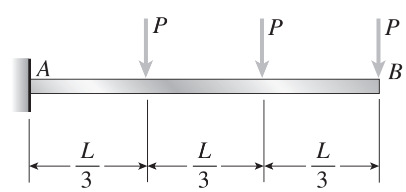
$$v' = -\frac{q_0}{5760LEI}[L^2(37L^2 - 120x^2) + 30(2x - L)^4]$$

$$\theta_B = v'(L) = \frac{53q_0L^3}{5760EI} \quad \leftarrow$$

Method of Superposition

The problems for Section 9.5 are to be solved by the method of superposition. All beams have constant flexural rigidity EI.

Problem 9.5-1 A cantilever beam AB carries three equally spaced concentrated loads, as shown in the figure. Obtain formulas for the angle of rotation θ_B and deflection δ_B at the free end of the beam.



Solution 9.5-1 Cantilever beam with 3 loads

Table G-1, Cases 4 and 5

$$\theta_B = \frac{P\left(\frac{L}{3}\right)^2}{2EI} + \frac{P\left(\frac{2L}{3}\right)^2}{2EI} + \frac{PL^2}{2EI} = \frac{7PL^2}{9EI} \quad \leftarrow$$

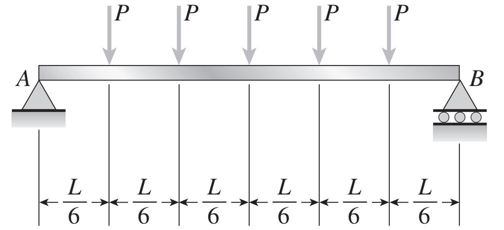
$$\delta_B = \frac{P\left(\frac{L}{3}\right)^2}{6EI}\left(3L - \frac{L}{3}\right) + \frac{P\left(\frac{2L}{3}\right)^2}{6EI}\left(3L - \frac{2L}{3}\right) + \frac{PL^3}{3EI} = \frac{5PL^3}{9EI} \quad \leftarrow$$

Problem 9.5-2 A simple beam *AB* supports five equally spaced loads *P* (see figure).

(a) Determine the deflection δ_1 at the midpoint of the beam.

(b) If the same total load (5*P*) is distributed as a uniform load on the beam, what is the deflection δ_2 at the midpoint?

(c) Calculate the ratio of δ_1 to δ_2.



Solution 9.5-2 **Simple beam with 5 loads**

(a) Table G-2, Cases 4 and 6

$$\delta_1 = \frac{P\left(\frac{L}{6}\right)}{24EI}\left[3L^2 - 4\left(\frac{L}{6}\right)^2\right] + \frac{P\left(\frac{L}{3}\right)}{24EI}\left[3L^2 - 4\left(\frac{L}{3}\right)^2\right] + \frac{PL^3}{48EI}$$

$$= \frac{11PL^3}{144EI} \quad \leftarrow$$

(b) Table G-2, Case 1 $qL = 5P$

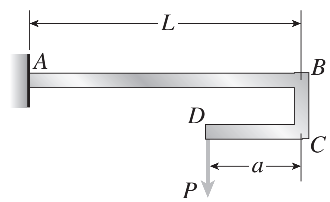
$$\delta_2 = \frac{5qL^4}{384EI} = \frac{25PL^3}{384EI} \quad \leftarrow$$

(c) $$\frac{\delta_1}{\delta_2} = \frac{11}{144}\left(\frac{384}{25}\right) = \frac{88}{75} = 1.173 \quad \leftarrow$$

Problem 9.5-3 The cantilever beam *AB* shown in the figure has an extension *BCD* attached to its free end. A force *P* acts at the end of the extension.

(a) Find the ratio *a*/*L* so that the vertical deflection of point *B* will be zero.

(b) Find the ratio *a*/*L* so that the angle of rotation at point *B* will be zero.



Solution 9.5-3 **Cantilever beam with extension**

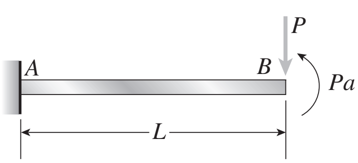


Table G-1, Cases 4 and 6

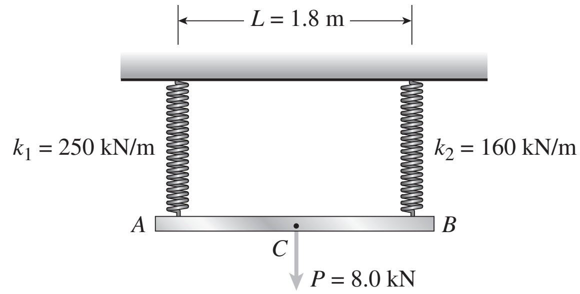
(a) $$\delta_B = \frac{PL^3}{3EI} - \frac{PaL^2}{2EI} = 0 \quad \frac{a}{L} = \frac{2}{3} \quad \leftarrow$$

(b) $$\theta_B = \frac{PL^2}{2EI} - \frac{PaL}{EI} = 0 \quad \frac{a}{L} = \frac{1}{2} \quad \leftarrow$$

Problem 9.5-4 Beam *ACB* hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity *EI*.

What is the downward displacement of point *C*, which is at the midpoint of the beam, when the load *P* is applied?

Data for the structure are as follows: $P = 8.0$ kN, $L = 1.8$ m, $EI = 216$ kN·m², $k_1 = 250$ kN/m, and $k_2 = 160$ kN/m.



Solution 9.5-4 Beam hanging from springs

$P = 8.0$ kN $\quad L = 1.8$ m

$EI = 216\ \text{kN}\cdot\text{m}^2 \quad k_1 = 250$ kN/m

$k_2 = 160$ kN/m

Stretch of springs:

$$\delta_A = \frac{P/2}{k_1} \qquad \delta_B = \frac{P/2}{k_2}$$

Table G-2, Case 4

$$\delta_C = \frac{PL^3}{48\,EI} + \frac{1}{2}\left(\frac{P/2}{k_1} + \frac{P/2}{k_2}\right)$$

$$= \frac{PL^3}{48EI} + \frac{P}{4}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \longleftarrow$$

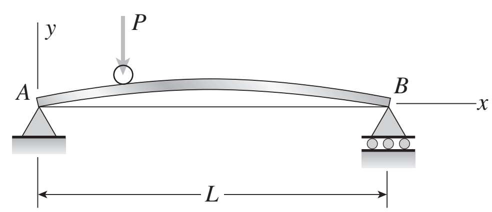
Substitute numerical values:

$$\delta_C = \frac{(8.0\ \text{kN})(1.8\ \text{m})^3}{48\,(216\ \text{kN}\cdot\text{m}^2)} + \frac{8.0\ \text{kN}}{4}\left(\frac{1}{250\ \text{kN/m}} + \frac{1}{160\ \text{kN/m}}\right)$$

$$= 4.5\ \text{mm} + 20.5\ \text{mm}$$

$$= 25\ \text{mm} \longleftarrow$$

Problem 9.5-5 What must be the equation $y = f(x)$ of the axis of the slightly curved beam *AB* (see figure) *before* the load is applied in order that the load *P*, moving along the bar, always stays at the same level?



Solution 9.5-5 Slightly curved beam

Let x = distance to load P

δ = downward deflection at load P

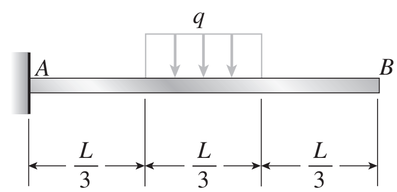
Table G-2, Case 5:

$$\delta = \frac{P(L-x)\,x}{6LEI}\,[L^2 - (L-x)^2 - x^2] = \frac{Px^2\,(L-x)^2}{3LEI}$$

Initial upward displacement of the beam must equal δ.

$$\therefore\ y = \frac{Px^2\,(L-x)^2}{3LEI} \longleftarrow$$

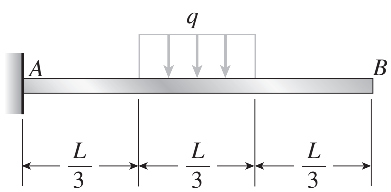
Problem 9.5-6 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB having a uniform load of intensity q acting over the middle third of its length (see figure).



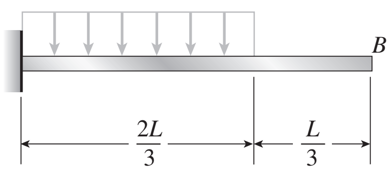
Solution 9.5-6 Cantilever beam (partial uniform load)

q = intensity of uniform load

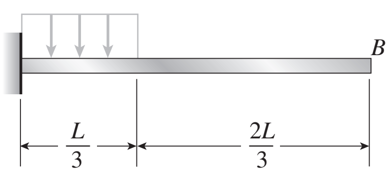
Original load on the beam:



Load No. 1:



Load No. 2:



SUPERPOSITION:

Original load = Load No. 1 minus Load No. 2

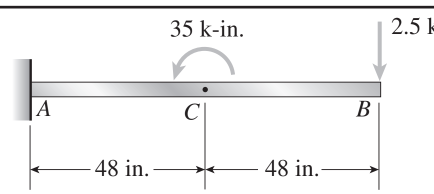
Table G-1, Case 2

$$\theta_B = \frac{q}{6EI}\left(\frac{2L}{3}\right)^3 - \frac{q}{6EI}\left(\frac{L}{3}\right)^3 = \frac{7qL^3}{162EI} \quad \longleftarrow$$

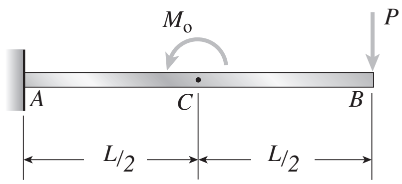
$$\delta_B = \frac{q}{24EI}\left(\frac{2L}{3}\right)^3\left(4L - \frac{2L}{3}\right) - \frac{q}{24EI}\left(\frac{L}{3}\right)^3\left(4L - \frac{L}{3}\right)$$

$$= \frac{23qL^4}{648EI} \quad \longleftarrow$$

Problem 9.5-7 The cantilever beam ACB shown in the figure has flexural rigidity $EI = 2.1 \times 10^6$ k-in.^{2} Calculate the downward deflections δ_C and δ_B at points C and B, respectively, due to the simultaneous action of the moment of 35 k-in. applied at point C and the concentrated load of 2.5 k applied at the free end B.



Solution 9.5-7 **Cantilever beam (two loads)**



$EI = 2.1 \times 10^6$ k-in.^{2}
$M_0 = 35$ k-in.
$P = 2.5$ k
$L = 96$ in.

Table G-1, Cases 4, 6, and 7

$$\delta_C = -\frac{M_0(L/2)^2}{2EI} + \frac{P(L/2)^2}{6EI}\left(3L - \frac{L}{2}\right)$$

$$= -\frac{M_0L^2}{8EI} + \frac{5PL^3}{48EI} \quad (+ = \text{downward deflection})$$

$$\delta_B = -\frac{M_0(L/2)}{2EI}\left(2L - \frac{L}{2}\right) + \frac{PL^3}{3EI}$$

$$= -\frac{3M_0L^2}{8EI} + \frac{PL^3}{3EI} \quad (+ = \text{downward deflection})$$

SUBSTITUTE NUMERICAL VALUES:

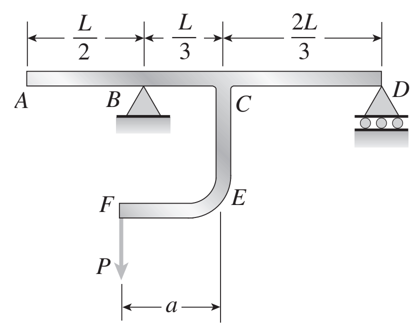
$\delta_C = -0.01920$ in. $+ 0.10971$ in.
$= 0.0905$ in. ←

$\delta_B = -0.05760$ in. $+ 0.35109$ in.
$= 0.293$ in. ←

Problem 9.5-8 A beam *ABCD* consisting of a simple span *BD* and an overhang *AB* is loaded by a force *P* acting at the end of the bracket *CEF* (see figure).

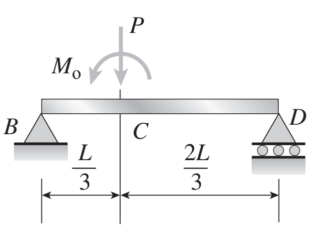
(a) Determine the deflection δ_A at the end of the overhang.

(b) Under what conditions is this deflection upward? Under what conditions is it downward?



Solution 9.5-8 **Beam with bracket and overhang**

Consider part *BD* of the beam.



$M_0 = Pa$

Table G-2, Cases 5 and 9

$$\theta_B = \frac{P(L/3)(2L/3)(5L/3)}{6LEI} + \frac{Pa}{6LEI}\left[6\left(\frac{L^2}{3}\right) - 3\left(\frac{L^2}{9}\right) - 2L^2\right]$$

$$= \frac{PL}{162EI}(10L - 9a) \quad (+ = \text{clockwise angle})$$

(a) DEFLECTION AT THE END OF THE OVERHANG

$$\delta_A = \theta_B\left(\frac{L}{2}\right) = \frac{PL^2}{324\,EI}(10L - 9a) \quad \leftarrow$$

$(+ = \text{upward deflection})$

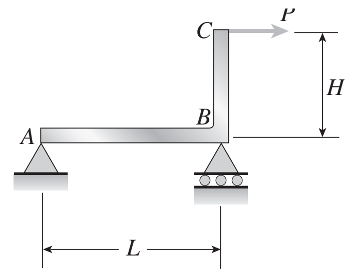
(b) Deflection is upward when $\frac{a}{L} < \frac{10}{9}$ and downward when $\frac{a}{L} > \frac{10}{9}$ ←

Problem 9.5-9 A horizontal load P acts at end C of the bracket ABC shown in the figure.

(a) Determine the deflection δ_C of point C.

(b) Determine the maximum upward deflection δ_{max} of member AB.

Note: Assume that the flexural rigidity EI is constant throughout the frame. Also, disregard the effects of axial deformations and consider only the effects of bending due to the load P.



Solution 9.5-9 Bracket *ABC*

BEAM AB

$M_0 = PH$

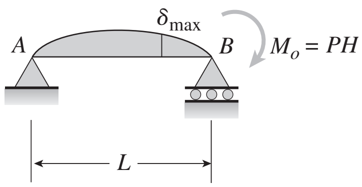


Table G-2, Case 7: $\theta_B = \dfrac{M_0 L}{3EI} = \dfrac{PHL}{3EI}$

(a) ARM BC Table G-1, Case 4

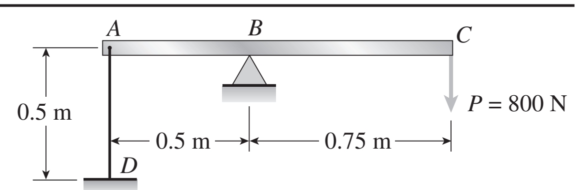
$$\delta_C = \frac{PH^3}{3EI} + \theta_B H = \frac{PH^3}{3EI} + \frac{PH^2L}{3EI}$$
$$= \frac{PH^2}{3EI}(L + H) \quad \longleftarrow$$

(b) MAXIMUM DEFLECTION OF BEAM AB

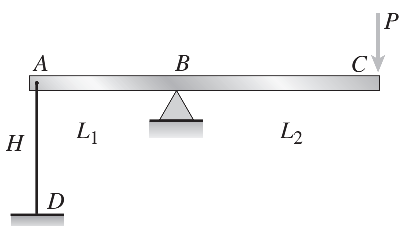
Table G-2, Case 7: $\delta_{max} = \dfrac{M_0 L^2}{9\sqrt{3}EI} = \dfrac{PHL^2}{9\sqrt{3}EI} \quad \longleftarrow$

Problem 9.5-10 A beam ABC having flexural rigidity $EI = 75\ \text{kN}\cdot\text{m}^2$ is loaded by a force $P = 800$ N at end C and tied down at end A by a wire having axial rigidity $EA = 900$ kN (see figure).

What is the deflection at point C when the load P is applied?



Solution 9.5-10 Beam tied down by a wire



$EI = 75\ \text{kN}\cdot\text{m}^2$

$P = 800$ N

$EA = 900$ kN

$H = 0.5$ m $\quad L_1 = 0.5$ m

$L_2 = 0.75$ m

CONSIDER BC AS A CANTILEVER BEAM

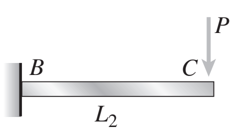
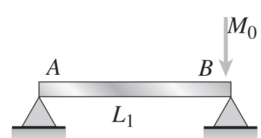


Table G-1, Case 4: $\delta'_C = \dfrac{PL_2^3}{3EI}$

CONSIDER AB AS A SIMPLE BEAM



$M_0 = PL_2$

Table G-2, Case 7: $\theta'_B = \dfrac{M_0 L_1}{3EI} = \dfrac{PL_1L_2}{3EI}$

CONSIDER THE STRETCHING OF WIRE AD

$$\delta'_A = (\text{Force in } AD)\left(\frac{H}{EA}\right) = \left(\frac{PL_2}{L_1}\right)\left(\frac{H}{EA}\right) = \frac{PL_2H}{EAL_1}$$

DEFLECTION δ_C OF POINT C

$$\delta_C = \delta'_C + \theta'_B(L_2) + \delta'_A\left(\frac{L_2}{L_1}\right)$$
$$= \frac{PL_2^3}{3EI} + \frac{PL_1L_2^2}{3EI} + \frac{PL_2^2H}{EAL_1^2} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\delta_C = 1.50\ \text{mm} + 1.00\ \text{mm} + 1.00\ \text{mm} = 3.50\ \text{mm} \quad \longleftarrow$